

Spinons and parafermions in fermion cosets

Daniel C. Cabra*

International Centre for Theoretical Physics, Strada Costiera 11, 34100 Trieste, Italy

We introduce a set of gauge invariant fermion fields in fermionic coset models and show that they play a very central role in the description of several Conformal Field Theories (CFT's). In particular we discuss the explicit realization of primaries and their OPE in unitary minimal models, parafermion fields in Z_k CFT's and that of spinon fields in $SU(N)_k$, $k = 1$ Wess-Zumino-Witten models (WZW) theories. The higher level case ($k > 1$) will be briefly discussed. Possible applications to QHE systems and spin-ladder systems are addressed.

Introduction

The purpose of this talk is to briefly review the fermionic coset description of some particular CFT's, whose relevance has shown up in particular in the construction of the primary operators in the minimal unitary models and the Z_k parafermion models, as well as in the identification of quasiparticle operators in $SU(N)_k$ WZW models [1,2]. We will particularly emphasize the realization of order-disorder (OD) algebras as well as the realization of quasi-particle operators with generalized statistics, where the gauge invariant fermion fields play a central role.

We want to point out that we will only briefly comment on some of the basics of CFT which are relevant for the discussion we want to pursue. For more details see for example ref. [3].

The so called coset construction was introduced by Goddard, Kent and Olive [4] as a way to obtain CFT's with Virasoro central extension less than unity, and can be used to build up many interesting conformal field theories [3]. Lagrangian realizations of these coset theories have been presented, both in the bosonic formulation [5], in terms of gauged WZW models and in the fermionic versions [6], in terms of constrained (gauged) fermions.

Here we will use the fermionic description of coset models, and will discuss essentially three cases: the models in the so called minimal unitary or FQS series [7], the Z_k parafermion models [8] and the $SU(N)_k$ WZW models [2,9] (from the point of view of fermion cosets). However, the present approach can be applied to the study of arbitrary coset models. (Details on the computations that lead to the results reviewed here are contained in refs. [10–13]).

In the case of minimal models we have shown that all the primaries can be constructed as certain fermion bilinears, and in the case of the Ising model, (the first model in the minimal unitary series), we have shown that the role of the gauge invariant fermions is crucial in the identification of the OD fields. This is the first case in which the relevance of gauge invariant fermions has shown up [10].

In ref. [11] we have shown how the construction of the OD operators in [10] can be generalized to construct all the primaries in Z_k parafermion models, and we used this construction to study the “thermal” perturbation of the system.

More recently [12], we have shown that the gauge invariant fermions realize in a natural way the so called “spinon” fields, which were shown to play a crucial role in the quasiparticle description of the Hilbert space of the $SU(N)_1$ WZW theories, as motivated by the underlying Yangian symmetry [14]. (The extension to higher level cases is in progress [13]). As a byproduct, we have also shown how to factorize the WZW primaries in holomorphic and anti-holomorphic parts, a problem that has received some attention recently [15].

*Investigador CONICET, Argentina.

On leave from Universidad Nacional de La Plata, Argentina.

Talk given at the International Seminar “Supersymmetry and Quantum Field Theory” dedicated to the memory of D.V. Volkov, Kharkov, Ukraine, 5-7 jan. 1997

All these results clearly suggest that the gauge invariant fermions (gif's) play an important role in the description of CFT's: in the case of $SU(N)_k$ WZW models, these gif's are precisely the quasiparticle operators, and both in the minimal unitary models and in the Z_k parafermion models, all the primaries are composites (bound states) of these gif's. Hence, there is a good chance that the construction of quasi-particle operators in terms of gif's pursued in refs. [12,13] could be extended to other CFT's. This construction could be of some interest in connection with the so called quasiparticle (or fermionic) representation of the characters for CFT's [16].

Another interesting point is that the quasiparticle operators constructed in ref. [13] for the higher level cases satisfy non-abelian braiding relations, and hence could play a role in the description of some QHE systems, such as the so called Haldane-Rezayi and Pfaffian states [17], where the elementary quasiparticles have non-abelian statistics.

Finally we want to mention the so called spin-ladder systems (see ref. [18], and references therein), which in the low energy regime can be studied as certain WZW models with interactions. Our approach could be also useful in the study of these systems, where many interesting effects have a non-perturbative origin.

An example: $SU(N)_1$ WZW theory as a fermion coset

Let us first set up our conventions and describe the approach in a simple example, such as the fermionic coset representation of the $SU(N)_1$ WZW theory [9].

The idea is to describe this model as a fermionic coset $U(N)/U(1)$, which is constructed starting with N free massless Dirac fermions and freezing the $U(1)$ charge by imposing

$$j_\mu |phys\rangle = 0, \quad (1)$$

where j_μ is the $U(1)$ fermionic current.

This is achieved in the path-integral by introducing a δ -functional as

$$\delta[j_\mu] = \int Da_\mu \exp\left(-\int d^2x \psi^\dagger \gamma_\mu \psi a_\mu\right). \quad (2)$$

The partition function of the $U(N)/U(1)$ model is then given by

$$\mathcal{Z}_{U(N)/U(1)} = \int D\psi^\dagger D\psi Da_\mu \exp\left(-\int d^2x \psi^\dagger (i\partial + i\not{a})\psi\right), \quad (3)$$

and is equivalent to the partition function of the $SU(N)_1$ WZW model in the sense that correlators of corresponding fields in the two theories coincide [9].

In order to write the partition function in a more manageable form, we perform the following change of variables [19]

$$\begin{aligned} a &= i(\partial u)u^{-1} & \bar{a} &= i(\partial \bar{u})\bar{u}^{-1}, \\ \psi_1 &= u\chi_1 & \psi_2^\dagger &= \chi_2^\dagger u^{-1}, \\ \psi_2 &= \bar{u}\chi_2 & \psi_1^\dagger &= \chi_1^\dagger \bar{u}^{-1}. \end{aligned} \quad (4)$$

where $\partial \equiv \frac{\partial}{\partial z}$, $\bar{\partial} \equiv \frac{\partial}{\partial \bar{z}}$ and $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$. The fields u and \bar{u} are parametrized in terms of real scalar fields as $h = \exp(-\phi - i\eta)$ and $\bar{h} = \exp(\phi - i\eta)$.

Taking into account the gauge fixing procedure and the Jacobians associated to (4) [19] one arrives at the desired decoupled form for the partition function:

$$\mathcal{Z} = \mathcal{Z}_{ff} \mathcal{Z}_{fb} \mathcal{Z}_{gh}, \quad (5)$$

where

$$\begin{aligned} \mathcal{Z}_{ff} &= \int \mathcal{D}\chi^\dagger \mathcal{D}\chi \exp\left(-\frac{1}{\pi} \int (\chi_2^\dagger \bar{\partial} \chi_1 + \chi_1^\dagger \partial \chi_2) d^2x\right), \\ \mathcal{Z}_{fb} &= \int \mathcal{D}\phi \exp\left(\frac{N}{2\pi} \int \phi \Delta \phi d^2x\right). \end{aligned} \quad (6)$$

(The explicit form of the ghost partition function is inessential in what follows).

Notice that, although the partition function of the theory is completely decoupled, BRST quantization conditions connect the different sectors in order to ensure unitarity [5].

The central charge is now easily evaluated as the sum of three independent contributions coming from the different sectors, $c_{ff} = N$, $c_{fb} = 1$ and $c_{gh} = -2$, thus giving $c = N - 1$ which coincides with the central charge of the $SU(N)_1$ WZW model. Similarly, conformal dimensions of primaries can be evaluated using this decoupled picture.

More general models, as for example the $SU(2)_k$ WZW models, can be also represented as fermionic cosets by making use of the general equivalence [9]

$$U(2k)/(U(1) \times SU(k)_2). \quad (7)$$

In this case, in addition to the constraint implemented by the abelian gauge field a_μ , we have to introduce another constraint associated with the $SU(k)$ currents. This constraint will be implemented by a non-abelian gauge field in the Lie algebra of $SU(k)$. The $SU(k)$ gauge field is traded for a $SU(k)$ WZW field through a decoupling transformation similar to that of eq.(4).

Using the approach described above we have studied the following cases:

i) Minimal unitary models

These models can be represented as the cosets [4]

$$(SU(2)_k \times SU(2)_1)/SU(2)_{k+1}, \quad (8)$$

and the Virasoro central charges lie in the FQS series [7]

$$c = 1 - 6/((k+2)(k+3)). \quad (9)$$

By making use of eq.(7), one is led to make the identification of the coset (8) with the fermion coset

$$\left[\left(\frac{U(2k)}{SU(k)_2 \times U(1)} \right) \times \left(\frac{U(2)}{U(1)} \right) \right] / SU(2)_{k+1}, \quad (10)$$

whose Lagrangian is given by

$$\mathcal{L} = \psi^\dagger (\partial + a + B + A) \psi + \bar{\chi}^\dagger (\partial + b + A) \chi, \quad (11)$$

where all the internal indices are suppressed. (The fermions ψ (χ) transform in the fundamental representation of $U(2k)$ ($U(2)$)). The gauge fields a_μ , b_μ , B_μ , A_μ , in the Lagrangian (11), implement respectively the $U(1)$, $U(1)$, $SU(k)_2$ and $SU(2)_{k+1}$. (The subindices refer to the central charge of the constrained affine currents).

Within this approach one can show that the central charge of these fermionic coset models is given by eq.(9) [6], all the primaries can be constructed in terms of fermion bilinears [10], and the correlators can be evaluated in terms of known results on WZW theories [20].

The point we want to make here is that the fermionic description is more suitable than the bosonic one regarding the construction of primary fields and their operator product algebra. In particular we can identify in a natural way the OD fields as we will show in the examples below. There is one interesting example that has been completely described within this scheme, which corresponds to the Ising model ($c = 1/2$, $k = 1$ in eq.(9)). In this case, not only the primary operators, (i.e. the fields with their dimensions in the Kac table), but also the OD algebra has been realized using the gauge invariant fermions [10].

Gauge invariant fermions

Let us introduce the gauge invariant fermion fields of the theory defined by eq.(11), for $k = 1$,

$$\hat{\psi}^i(x) = e^{-i \int_x^\infty dz_\mu a_\mu} \left(P e^{-i \int_x^\infty dz^\mu A_\mu} \right)_{ij} \psi^j(x) \quad (12)$$

$$\hat{\chi}^i(x) = e^{-i \int_x^\infty dz_\mu b_\mu} \left(P e^{-i \int_x^\infty dz^\mu A_\mu} \right)_{ij} \chi^j(x). \quad (13)$$

(Note that for $k = 1$, the B field is not present)

In terms of these fields, the spin (σ) and disorder (μ) fields read

$$\sigma(x) = \hat{\psi}^\dagger \hat{\psi} + \hat{\chi}^\dagger \hat{\chi}, \quad \text{and} \quad \mu(x) = \hat{\psi}^\dagger \hat{\chi} + \hat{\chi}^\dagger \hat{\psi}. \quad (14)$$

Using the decoupling formulation one can show that they have the correct dimensions (i.e. $h = \bar{h} = 1/16$), and satisfy the OD algebra [21]

$$\sigma(z_1)\mu(z_2) = e^{i\pi\Theta(z_1-z_2)}\mu(z_2)\sigma(z_1), \quad (15)$$

where $\Theta(z)$ stands for the Heaviside function.

We will explain how the OD algebra arises in a simpler example later on, but let us point out that while the Schwinger line integrals associated with the non-abelian gauge field A cancel out in the bilinears (14), those associated with the abelian gauge fields a and b will not, and this fact will give rise to the OD algebra.

ii) Z_k parafermion models

These models are realized as the cosets

$$SU(2)_k/U(1), \quad (16)$$

and the Virasoro central charge is given by $c = 2(k-1)/(k+2)$.

In terms of fermionic cosets they correspond to

$$\left(\frac{U(2k)}{U(1) \times SU(k)_2} \right) / U(1), \quad (17)$$

In order to construct the primary fields we have proceeded in two steps: first we have used the identification of the primary fields of the $SU(2)_k$ WZW in terms of fermion bilinears made in ref. [9], and then, rewriting these bilinears in terms of gif's we have shown that they correspond to OD fields. Other primaries of the Z_k models, such as parafermion currents, were also constructed as suitable composites of gauge invariant fermions.

$SU(2)_k$ primaries

The Lagrangian of the fermionic description of the $SU(2)_k$ WZW model is given by

$$\mathcal{L} = \psi^\dagger (\not{\partial} + \not{a} + \not{B}) \psi, \quad (18)$$

where a_μ and B_μ are $U(1)$ and $SU(k)$ gauge fields, implementing the respective constraints.

The fundamental field g and its adjoint g^\dagger of the bosonic $SU(2)_k$ WZW theory are represented in terms of fermions by the bosonization formulae [9]

$$g^{ij} = \psi_2^i \psi_2^{j\dagger}, \quad g^{ij\dagger} = \psi_1^i \psi_1^{j\dagger}. \quad (19)$$

All other integrable representations can be constructed as symmetrized normal ordered products of these fundamental fields. Note that these fields are invariant under gauge transformations, both in $U(1)$ and $SU(k)$, as they should be in order to correspond to physical operators.

The theory which we are interested in has one more $U(1)$ constraint (see eq.(17)), which we implement by adding to the Lagrangian (18) the term

$$\Delta L = \psi^\dagger b^3 t^3 \psi, \quad (20)$$

where t^3 is an $SU(2)$ generator. This term (after functional integration over b_μ^3) implements the additional $U(1)$ constraint as follows from eqs.(1),(2). It is easy to show, using the approach described above, that the central charge of the resulting model is the correct one.

The fields in (19) vary under gauge transformations associated with the new gauge field, b_μ^3 , introduced in eq.(20). In order to ensure invariance also under these transformations, we will define the gauge invariant fermions as

$$\hat{\psi} = e^{-i \int_x^\infty dz^\mu b_\mu^3} t^3 \psi. \quad (21)$$

In [11] we have shown that all the Z_k primaries can be built up from these fields. Here we will only discuss an example in some detail, in order to show how OD algebras are realized in terms of gifs.

Using (21) we can construct the gauge invariant version of the g -field and its adjoint in eq.(19) [10]

$$\hat{g}_{ij} = \hat{\psi}_2^i \hat{\psi}_2^{j\dagger}, \quad \hat{g}_{ij}^\dagger = \hat{\psi}_1^i \hat{\psi}_1^{j\dagger}. \quad (22)$$

Let us consider the composites

$$\sigma_1 \equiv \hat{g}_{1,1}, \quad \mu_1 \equiv \hat{g}_{2,1}^\dagger. \quad (23)$$

In the decoupled picture these fields can be rewritten as

$$\begin{aligned} \sigma_1 \equiv \hat{g}_{1,1} &= : \chi_2^{1\alpha} \tilde{U}^{-1}{}^{\alpha\beta} \chi_2^{\dagger 1\beta} :: e^{2\phi_a} :: e^{2\phi_b} :, \\ \mu_1 \equiv \hat{g}_{2,1}^\dagger &= : \chi_1^{2\alpha} \tilde{U}^{\alpha\beta} \chi_1^{\dagger 1\beta} :: e^{-2\phi_a} :: e^{\varphi_b - \bar{\varphi}_b} :. \end{aligned} \quad (24)$$

Here \tilde{U} is the $SU(k)$ WZW field that parametrizes the field B_μ , and ϕ_a, ϕ_b are the $U(1)$ boson fields that parametrize a_μ and b_μ respectively, and φ_b and $\bar{\varphi}_b$ are the chiral (holomorphic and anti-holomorphic) components of the free boson ϕ_b .

The dimensions of these fields are easily evaluated in the decoupled picture and are given by

$$h = \bar{h} = \frac{k-1}{2k(k+2)} \quad (25)$$

and it can be shown that they satisfy the OD algebra [10]

$$\sigma_1(x_1) \mu_1(x_2) = e^{\frac{i2\pi}{k} \Theta(x_1 - x_2)} \mu_1(x_2) \sigma_1(x_1). \quad (26)$$

Eqs. (25) and (26) lead one to identify the fields σ and μ defined in (23) with the order and disorder operators in the Z_k parafermion theory.

Let us stress that the OD algebra has its origin in the particular way in which the holomorphic components of the free boson ϕ_b are combined in eq.(24). This, in turn, is a consequence of the use of the gauge invariant fermions (21). Indeed, one can easily check, using the canonical commutation rules for ϕ_b that

$$: e^{2\phi_b(x_1)} :: e^{(-\varphi_b + \bar{\varphi}_b)(x_2)} := e^{\frac{i2\pi}{k} \Theta(x_1 - x_2)} : e^{(-\varphi_b + \bar{\varphi}_b)(x_2)} :: e^{2\phi_b(x_1)} :, \quad (27)$$

being the other factors commuting.

This construction can be generalized to all other Z_k primaries and hence, having identified all the primaries one can pursue the study of perturbations in the Lagrangian approach. We have studied the “thermal” perturbation of Z_k models within the present approach and reduced the problem to the study of interacting WZW fields. In this scheme, two fixed points of the perturbed system were identified [11].

iii) $SU(N)_k$ WZW models: gif’s, spinons and holomorphic factorization

All we have done so far is to study bilinears of the gauge invariant fermions. In this last section we will study the properties of the gauge invariant fermions themselves and show their relevance in the so called “spinon” description of WZW models [14], as described in ref. [12,13].

Let us then define the gauge invariant fermion fields [12] corresponding to the $SU(N)_k$ theory similarly as in (12), (with A_μ replaced by B_μ), that by construction will create the physical excitations, and let us study their properties. For simplicity we will discuss the level one case, although similar considerations apply to the higher level case [13].

In the decoupled picture, the gifs are given by

$$\begin{aligned} \hat{\psi}_1^i(z) &= e^{-\varphi(z)} \chi_1^i(z) & \hat{\psi}_1^{i\dagger}(\bar{z}) &= \chi_1^{i\dagger}(\bar{z}) e^{-\bar{\varphi}(\bar{z})} \\ \hat{\psi}_2^i(\bar{z}) &= e^{\bar{\varphi}(\bar{z})} \chi_2^i(\bar{z}) & \hat{\psi}_2^{i\dagger}(z) &= \chi_2^{i\dagger}(z) e^{\varphi(z)} \end{aligned} \quad (28)$$

where $\varphi(z)$ and $\bar{\varphi}$ are the chiral (holomorphic and anti-holomorphic) components of the free boson ϕ . This fact together with the equation of motion of the free fermions χ ensures that $\hat{\psi}_1^i$ and $\hat{\psi}_2^{i\dagger}$ ($\hat{\psi}_2^i$ and $\hat{\psi}_1^{i\dagger}$) are holomorphic

(anti-holomorphic). The conformal dimensions are given by $((N-1)/2N, 0)$ and $(0, (N-1)/2N)$ respectively, thus suggesting that they correspond to the “halves” of the WZW primary $g(z, \bar{z})$.

In fact, it can be shown by the following sequence of identities

$$g^{ij}(z, \bar{z}) = \psi_2^i \psi_2^{j\dagger} = \hat{\psi}_2^i(\bar{z}) \hat{\psi}_2^{j\dagger}(z), \quad (29)$$

that the gifs are the holomorphic factors of the WZW primary.

The second important property satisfied by these operators is their OPE which allows us to identify them as the “spinon” operators of ref. [14]:

$$\hat{\psi}_1^i(z) \hat{\psi}_1^j(w) = \frac{1}{(z-w)^{1/N}} \mathcal{A}(\chi_1^i(w) \chi_1^j(w)) : \exp 2\varphi(w) : + \dots = \frac{1}{(z-w)^{1/N}} \Phi_2^{ij}(w) + \dots, \quad (30)$$

where \mathcal{A} stands for antisymmetrization and Φ_2^{ij} is the WZW primary with dimension $(N-2)/N$.

As explained in the third reference of [14] the branch cut in the OPE singularity implies that the excitations created by the spinon fields satisfy generalized statistics with “statistical angle” $\theta = \pi/N$. The chiral Fock space of the $SU(N)_1$ CFT can be constructed in terms of the modes of these fields and this space can be classified into multiplets corresponding to the irreducible representations of the Yangian algebra $Y(sl_N)$ [14].

In the case of higher levels, the same construction can be done, and in this case, the excitations created by the gauge invariant fermions satisfy non-abelian braiding statistics. The holomorphic factorization of the WZW fields in terms of gifs can also be proved. The results of this investigation will appear elsewhere [13].

Acknowledgements: I would like to thank to my collaborators E. Moreno, G. Rossini and K.D.Rothe with whom the work presented in this talk has been developed. This work was supported in part by CONICET and Fundación Antorchas, Argentina.

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